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COALGEBRA-GALOIS EXTENSIONS FROM THE EXTENSION THEORY POINT OF VIEW

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Abstract. Coalgebra-Galois extensions generalise Hopf-Galois extensions, which can be viewed as non-commutative torsors. In this paper it is analysed when a coalgebra-Galois extension is a separable, split, or strongly separable extension.

1. Introduction

Given a coalgebra C, an algebra A and a right coaction $\rho^A : A \to A \otimes C$ one can define a fixed point subalgebra B of A as consisting of all those elements of A over which the coaction is left-linear. In this way one obtains an extension $B \hookrightarrow A$, which is called a *coalgebra-Galois extension* if a certain canonical left A-module, right C-comodule map is bijective [4] [3]. The aim of this article is to analyse such coalgebra-Galois extensions from the extension theory point of view. In particular we study the problem when such extensions are separable, split or strongly separable extensions. This problem is put in a broader context of *entwining structures* and *entwined modules* introduced in [4] [2], as a generalisation of a Doi-Hopf datum and Doi-Koppinen modules [10] [15], respectively. We make use of the notion of a separability of a functor introduced in [17], and, as a byproduct, we generalise some of the results of [5] obtained recently for Doi-Koppinen modules.

The paper is organised as follows. In Section 2 we recall definitions and give examples of entwining structures and entwined modules. In Section 3 we analyse when certain functors between categories of entwined modules induced by morphisms of entwining structures are separable. In Section 4 we apply the results of Section 3 to prove that a sufficient and necessary condition for a coalgebra-Galois extension to be separable is the separability of a certain induction functor. This, in turn, is equivalent to the existence of a normalised integral in the canonical entwining structure. In Section 5 we analyse when a coalgebra-Galois extension is a split extension. This turns out to be related to the separability of the forgetful functor from the category of entwined modules to the category of right modules - another special case of the main theorem in Section 3. Finally, in Section 6 we study the problem when a coalgebra-Galois extension is a strongly separable extension in the sense of [14].

We work over a commutative ring k with identity 1. We assume that all the algebras are over k, associative and unital, and the coalgebras are over k, coassociative and counital. Unadorned tensor product is over k. For any k-modules V, W the symbol $\mathrm{Hom}(V, W)$ denotes the k-module of k-linear maps $V \to W$ and the identity map $V \to V$ is denoted by V. The twist map between k-modules V, W is denoted

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by twist : $V\otimes W\to W\otimes V$, $v\otimes w\mapsto w\otimes v$. We also implicitly identify V with $V\otimes k$ and $k\otimes V$ via the canonical isomorphisms.

For a k-algebra A we use μ_A to denote the product as a map and 1_A to denote the identity both as an element of A and as a map $k\to A$, $\alpha\mapsto\alpha 1_A$. $\mathbf{M}_A$ (resp. ${}_A\mathbf{M}$) denotes the category of right (resp. left) A-modules. The morphisms in this category are denoted by $\mathrm{Hom}_A(M,N)$ (resp. ${}_A\mathrm{Hom}(M,N)$). For any $M\in\mathbf{M}_A$ (resp. $M\in{}_A\mathbf{M}$), the symbol ρ_M (resp. ${}_M\rho$) denotes the action as a map (on elements the action is denoted by a dot). We often write M_A (resp. ${}_AM$) to indicate in which context the A-module M appears. For any $M\in\mathbf{M}^A$, $N\in{}^A\mathbf{M}$ we will write $\mathrm{eq}_{M_AN}:M\otimes A\otimes N\to M\otimes N$ for the action equalising map defining tensor product $M\otimes_A N$, i.e., $\mathrm{eq}_{M_AN}=\rho_M\otimes N-M\otimes{}_N\rho$, $M\otimes_A N=\mathrm{coker}(\mathrm{eq}_{M_AN})$.

For a k-coalgebra C we use Δ_C to denote the coproduct and ϵ_C to denote the counit. Notation for comodules is similar to that for modules but with subscripts replaced by superscripts, i.e. $\mathbf{M}^C$ is the category of right C-comodules, ρ^M is a right coaction etc. We use the Sweedler notation for coproducts and coactions, i.e. $\Delta_C(c)=c_{(1)}\otimes c_{(2)}$, $\rho^M(m)=m_{(0)}\otimes m_{(1)}$ (summation understood). For any $V\in\mathbf{M}^C$, $W\in{}^C\mathbf{M}$, $V\Box_CW$ denotes the cotensor product, which is defined by the exact sequence

$$0\longrightarrow V\Box_CW\longrightarrow V\otimes W\xrightarrow{\mathrm{eq}^{V^CW}}V\otimes C\otimes W,$$

where eq^{V^CW} is the coaction equalising map, i.e., $\mathrm{eq}^{V^CW}=\rho^V\otimes W-V\otimes{}^W\rho$.

2. Preliminaries on entwining structures and coalgebra-Galois extensions

Definition 2.1. An *entwining structure* (over k) is a triple $(A,C)_\psi$ consisting of a k-algebra A, a k-coalgebra C and a k-linear map $\psi:C\otimes A\to A\otimes C$ satisfying

$$\psi\circ(C\otimes\mu_A)=(\mu_A\otimes C)\circ(A\otimes\psi)\circ(\psi\otimes A),\quad \psi\circ(C\otimes 1_A)=1_A\otimes C,\tag{1}$$

$$(A\otimes\Delta_C)\circ\psi=(\psi\otimes C)\circ(C\otimes\psi)\circ(\Delta_C\otimes A),\quad (A\otimes\epsilon_C)\circ\psi=\epsilon_C\otimes A.\tag{2}$$

A morphism of entwining structures is a pair $(f,g):(A,C)_\psi\to(\tilde{A},\tilde{C})_{\tilde{\psi}}$, where $f:A\to\tilde{A}$ is an algebra map, $g:C\to\tilde{C}$ is a coalgebra map, and $(f\otimes g)\circ\psi=\tilde{\psi}\circ(g\otimes f)$.

The category of entwining structures is a tensor category with tensor product $(A,C)_\psi\otimes(\tilde{A},\tilde{C})_{\tilde{\psi}}=(A\otimes\tilde{A},C\otimes\tilde{C})_{(A\otimes\mathrm{twist}\otimes\tilde{C})\circ(\psi\otimes\tilde{\psi})\circ(C\otimes\mathrm{twist}\otimes\tilde{A})}$, and unit object $(k,k)_{\mathrm{twist}}$.

For $(A,C)_\psi$ we use the notation $\psi(c\otimes a)=a_\alpha\otimes c^\alpha$ (summation over a Greek index understood), for all $a\in A$, $c\in C$. The notion of an entwining structure was introduced in [4, Definition 2.1]. It is self-dual in the sense that conditions in Definition 2.1 are invariant under the operation consisting of interchanging of A with C, μ_A with Δ_C, and 1_A with ϵ_C, and reversing the order of maps. Below are two classes of examples of entwining structures coming from Galois-extensions.

Example 2.2 ([3])**.** Let C be a coalgebra, A an algebra and a right C-comodule. Let $B:=\{b\in A\mid\rho^A(ba)=b\rho^A(a)\}$ and assume that the canonical left A-module, right C-comodule map $\mathrm{can}:A\otimes_BA\to A\otimes C$, $a\otimes a'\mapsto a\rho^A(a')$, is bijective. Let $\psi:C\otimes A\to A\otimes C$ be a k-linear map given by $\psi(c\otimes a)=\mathrm{can}(\mathrm{can}^{-1}(1_A\otimes c)a)$.

Then $(A,C)_\psi$ is an entwining structure. The extension $B\hookrightarrow A$ is called a *coalgebra-Galois extension* (or a *C-Galois extension*) and is denoted by $A(B)^C$. $(A,C)_\psi$ is the *canonical entwining structure* associated to $A(B)^C$. A coalgebra-Galois extension $A(B)^C$ is said to be *copointed* if there exists a group-like $e\in C$ such that $\rho^A(1_A)=1_A\otimes e$.

Dually we have

Example 2.3 ([3])**.** Let A be an algebra, C a coalgebra and a right A-module. Let $B:=C/I$, where I is a coideal in C,

$$I:=\mathrm{span}\{(c\cdot a)_{(1)}\xi((c\cdot a)_{(2)})-c_{(1)}\xi(c_{(2)}\cdot a)\mid a\in A,\ c\in C,\ \xi\in C^*\},$$

and assume that the canonical left C-comodule, right A-module map cocan : $C\otimes A\to C\Box_B C$, $c\otimes a\mapsto c_{(1)}\otimes c_{(2)}\cdot a$, is bijective. Let $\psi:C\otimes A\to A\otimes C$ be a k-linear map given by $\psi=(\epsilon_C\otimes A\otimes C)\circ(\mathrm{cocan}^{-1}\otimes C)\circ(C\otimes\Delta_C)\circ\mathrm{cocan}$. Then $(A,C)_\psi$ is an entwining structure. The coextension $C\twoheadrightarrow B$ is called an *algebra-Galois coextension* (or an *A-Galois coextension*) and is denoted by $C(B)_A$. $(A,C)_\psi$ is the *canonical entwining structure* associated to $C(B)_A$. An algebra-Galois coextension $C(B)_A$ is said to be *pointed* if there exists an algebra map $\kappa:A\to k$ such that $\epsilon_C\circ\rho_C=\epsilon_C\otimes\kappa$.

Associated to an entwining structure is the category of entwined modules.

Definition 2.4. Let $(A,C)_\psi$ be an entwining structure. An (entwined) *$(A,C)_\psi$-module* is a right A-module, right C-comodule M such that

$$\rho^M\circ\rho_M=(\rho_M\otimes C)\circ(M\otimes\psi)\circ(\rho^M\otimes A),$$

(explicitly: $\rho^M(m\cdot a)=m_{(0)}\cdot a_\alpha\otimes m_{(1)}{}^\alpha$, $\forall a\in A, m\in M$). A morphism of $(A,C)_\psi$-modules is a right A-module map which is also a right C-comodule map. The category of $(A,C)_\psi$-modules is denoted by $\mathbf{M}_A^C(\psi)$.

The category $\mathbf{M}_A^C(\psi)$ was introduced and studied in [2]. An example of such modules are Doi-Koppinen modules introduced in [10], [15]. In this paper we will be concerned with two covariant functors between categories of entwined modules, which are special cases of the construction in [2, Section 3][1] (see also [7] for the Doi-Koppinen case). These functors are induced by certain morphisms of entwining structures.

Definition 2.5. Let $(f,g):(A,C)_\psi\to(\tilde A,\tilde C)_{\tilde\psi}$ be a morphism of entwining structures. View C as a left $\tilde C$-comodule via ${}^C\rho=(g\otimes C)\circ\Delta_C$ and $C\otimes\tilde A$ as a right $\tilde C$-comodule via $\rho^{C\otimes\tilde A}=(C\otimes\tilde\psi)\circ(C\otimes g\otimes\tilde A)\circ(\Delta_C\otimes\tilde A)$. Then (f,g) is said to be an *admissible morphism* iff:

(i) for all $\tilde M\in\mathbf{M}_{\tilde A}^{\tilde C}(\tilde\psi)$, $\tilde M\Box_{\tilde C}(C\otimes C)=(\tilde M\Box_{\tilde C}C)\otimes C$,

(ii) for all $M\in\mathbf{M}_A$, $(M\otimes(C\otimes\tilde A))\Box_{\tilde C}C=M\otimes((C\otimes\tilde A)\Box_{\tilde C}C)$.

For example, if $C,\tilde C$ are k-flat then (f,g) is an admissible morphism provided that ${}^{\tilde C}C$ is coflat. On the other hand if k is a regular ring or a field every morphism is admissible. Also, it can be easily checked that the following morphisms (A,ϵ_C) : $(A,C)_\psi\to(A,k)_{\mathrm{twist}}$ and $(1_A,C):(k,C)_{\mathrm{twist}}\to(A,C)_\psi$ are admissible.

[1] Although the paper [2] is restricted to k being a field, all the results of [2] quoted in the present paper can easily be seen to hold for a general k.

Example 2.6. Let $(f,g):(A,C)_\psi\to(\tilde{A},\tilde{C})_{\tilde{\psi}}$ be an admissible morphism of entwining structures. View $\tilde{A}$ as a right A-module via $\rho_{\tilde{A}}=\mu_{\tilde{A}}\circ(\tilde{A}\otimes f)$, and C as a right $\tilde{C}$-comodule via $\rho^C=(C\otimes g)\circ\Delta_C$. Then:

(1) For any $\tilde{M}\in\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$, $\tilde{M}\Box_{\tilde{C}}C$ is an $(A,C)_\psi$-module with structure maps $\rho^{\tilde{M}\Box_{\tilde{C}}C}=\tilde{M}\otimes\Delta_C$ and

$$\rho_{\tilde{M}\Box_{\tilde{C}}C}:\tilde{M}\Box_{\tilde{C}}C\otimes A\to\tilde{M}\Box_{\tilde{C}}C,\qquad \sum_i\tilde{m}_i\otimes c_i\otimes a=\sum_i\tilde{m}_i f(a_\alpha)\otimes c_i^\alpha.$$

(2) For any $M\in\mathbf{M}_A^C(\psi)$, $M\otimes_A\tilde{A}$ is an $(\tilde{A},\tilde{C})_{\tilde{\psi}}$-module with structure maps $\rho_{M\otimes_A\tilde{A}}=M\otimes_A\mu_{\tilde{A}}$ and

$$\rho^{M\otimes_A\tilde{A}}:M\otimes_A\tilde{A}\to M\otimes_A\tilde{A}\otimes\tilde{C},\qquad m\otimes\tilde{a}\mapsto m_{(0)}\otimes\tilde{a}_\alpha\otimes g(m_{(1)})^\alpha.$$

(3) The covariant functor $-\Box_{\tilde{C}}C:\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})\to\mathbf{M}_A^C(\psi)$ is the right adjoint of $-\otimes_A\tilde{A}:\mathbf{M}_A^C(\psi)\to\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$. The adjunctions are:

$$\forall M\in\mathbf{M}_A^C(\psi),\qquad \Phi_M:M\to(M\otimes_A\tilde{A})\Box_{\tilde{C}}C,\qquad m\mapsto m_{(0)}\otimes 1_{\tilde{A}}\otimes m_{(1)},$$

$$\forall\tilde{M}\in\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi}),\quad \Psi_{\tilde{M}}:(\tilde{M}\Box_{\tilde{C}}C)\otimes_A\tilde{A}\to\tilde{M},\quad \sum_i\tilde{m}_i\otimes c_i\otimes\tilde{a}\mapsto\sum_i\tilde{m}_i\cdot\tilde{a}\epsilon_C(c_i).$$

Applying Example 2.6 to morphisms $(A,\epsilon_C):(A,C)_\psi\to(A,k)_{\rm twist}$ and $(1_A,C):(k,C)_{\rm twist}\to(A,C)_\psi$ one obtains

Example 2.7. Let $(A,C)_\psi$ be an entwining structure. Then

(1) If M is a right A-module then $M\otimes C$ is an $(A,C)_\psi$-module with the coaction $M\otimes\Delta_C$ and the action $(m\otimes c)\cdot a=m\cdot\psi(c\otimes a)$, for all $a\in A, c\in C$ and $m\in M$. In particular $A\otimes C\in\mathbf{M}_A^C(\psi)$. The operation $M\mapsto M\otimes C$ defines a covariant functor $-\otimes C:\mathbf{M}_A\to\mathbf{M}_A^C(\psi)$ which is the right adjoint of the forgetful functor $\mathbf{M}_A^C(\psi)\to\mathbf{M}_A$.

(2) If V is a right C-comodule then $V\otimes A\in\mathbf{M}_A^C(\psi)$ with the action $V\otimes\mu_A$ and the coaction $v\otimes a\mapsto v_{(0)}\otimes\psi(v_{(1)}\otimes a)$ for any $a\in A$ and $v\in V$. In particular $C\otimes A\in\mathbf{M}_A^C(\psi)$. The operation $V\mapsto V\otimes A$ defines a covariant functor $-\otimes C:\mathbf{M}^C\to\mathbf{M}_A^C(\psi)$, which is the left adjoint of the forgetful functor $\mathbf{M}_A^C(\psi)\to\mathbf{M}^C$.

Another class of examples of entwined modules comes from (co)algebra-Galois (co)extensions [3]

Example 2.8.

(1) Let $(A,C)_\psi$ be the canonical entwining structure associated to a coalgebra-Galois extension $A(B)^C$. Then A is an $(A,C)_\psi$-module via ρ^A and μ_A.

(2) Let $(A,C)_\psi$ be the canonical entwining structure associated to an algebra-Galois coextension $C(B)_A$. Then C is an $(A,C)_\psi$-module via ρ_C and Δ_C.

3. Separable functors of entwined modules

In this section we analyse when functors described in Example 2.6 are separable. Recall from [17] that a covariant functor $F:\mathcal{C}\to\mathcal{D}$ is *separable* if the natural transformation ${\rm Hom}_{\mathcal{C}}(-,-)\to{\rm Hom}_{\mathcal{D}}(F(-),F(-))$ splits. In this paper we are dealing with the pairs of adjoint functors, so that the following characterisation of separable functors, obtained in [20] [8], is of great importance

Theorem 3.1. *Let $G : \mathcal{D} \to \mathcal{C}$ be the right adjoint of $F : \mathcal{C} \to \mathcal{D}$ with adjunctions $\Phi : 1_{\mathcal{C}} \to GF$ and $\Psi : FG \to 1_{\mathcal{D}}$. Then*

(1) F is separable if and only if Φ splits, i.e., for all objects $C \in \mathcal{C}$ there exists a morphism $\nu_C \in \mathrm{Mor}_{\mathcal{C}}(GF(C), C)$ such that $\nu_C \circ \Phi_C = C$ and for all $f \in \mathrm{Mor}_{\mathcal{C}}(C, \tilde{C})$, $\nu_{\tilde{C}} \circ GF(f) = f \circ \nu_C$.

(2) G is separable if and only if Ψ cosplits, i.e., for all objects $D \in \mathcal{D}$ there exists a morphism $\nu_D \in \mathrm{Mor}_{\mathcal{D}}(D, FG(D))$ such that $\Psi_D \circ \nu_D = D$ and for all $f \in \mathrm{Mor}_{\mathcal{D}}(D, \tilde{D})$, $\nu_{\tilde{D}} \circ f = FG(f) \circ \nu_D$.

Definition 3.2. An admissible morphism $(f, g) : (A, C)_\psi \to (\tilde{A}, \tilde{C})_{\tilde{\psi}}$ of entwining structures is said to be:

(1) *integrable* if there exists $\lambda \in \mathrm{Hom}_A((C \otimes \tilde{A})\Box_{\tilde{C}} C, A)$ such that the following diagrams commute

$$\begin{array}{ccccc}
(C \otimes A \otimes \tilde{A})\Box_{\tilde{C}} C & \xrightarrow{\psi\otimes\tilde{A}\otimes C} & A \otimes (C \otimes \tilde{A})\Box_{\tilde{C}} C & \xrightarrow{A\otimes\lambda} & A \otimes A \\
\downarrow{\scriptstyle C\otimes f\otimes\tilde{A}\otimes C} & & & & \downarrow{\scriptstyle \mu_A} \\
(C \otimes \tilde{A} \otimes \tilde{A})\Box_{\tilde{C}} C & \xrightarrow{C\otimes\mu_{\tilde{A}}\otimes C} & (C \otimes \tilde{A})\Box_{\tilde{C}} C & \xrightarrow{\lambda} & A,
\end{array} \qquad (3)$$

$$\begin{array}{ccccc}
(C \otimes \tilde{A})\Box_{\tilde{C}} C & \xrightarrow{C\otimes\tilde{A}\otimes\Delta_C} & (C \otimes \tilde{A})\Box_{\tilde{C}} C \otimes C & \xrightarrow{\lambda\otimes C} & A \otimes C \\
\downarrow{\scriptstyle \Delta_C\otimes\tilde{A}\otimes C} & & & & \| \\
C \otimes (C \otimes \tilde{A})\Box_{\tilde{C}} C & \xrightarrow{C\otimes\lambda} & C \otimes A & \xrightarrow{\psi} & A \otimes C.
\end{array} \qquad (4)$$

The right A-module structure of $(C \otimes \tilde{A})\Box_{\tilde{C}} C$ is as in Example 2.6(1), explicitly $\rho_{(C\otimes\tilde{A})\Box_{\tilde{C}} C} : c' \otimes \tilde{a} \otimes c \otimes a \mapsto c' \otimes \tilde{a} f(a_\alpha) \otimes c^\alpha$.

(2) *totally integrable*, if there exists $\lambda \in \mathrm{Hom}_A((C \otimes \tilde{A})\Box_{\tilde{C}} C, A)$ making it an integrable morphism and such that the following diagram

$$\begin{array}{ccc}
C & \xrightarrow{\Delta_C} & C \otimes C \\
\downarrow{\scriptstyle 1_A\circ\epsilon_C} & & \downarrow{\scriptstyle C\otimes 1_{\tilde{A}}\otimes C} \\
A & \xleftarrow{\lambda} & (C \otimes \tilde{A})\Box_{\tilde{C}} C
\end{array} \qquad (5)$$

commutes.

Notice that the condition (3) makes sense because ψ is a morphism in $\mathbf{M}_A^C(\psi)$, (f, g) is admissible and $(C\otimes\mu_{\tilde{A}})\circ(C\otimes f\otimes\tilde{A}) : C\otimes A\otimes\tilde{A} \to C\otimes\tilde{A}$ is a left $\tilde{C}$-comodule map, where the k-modules involved are left $\tilde{C}$-comodules via $(g \otimes C) \circ \Delta_C \otimes A \otimes \tilde{A}$ and $(g \otimes C) \circ \Delta_C \otimes \tilde{A}$, respectively. Similarly, condition (4) makes sense because Δ_C is a left $\tilde{C}$-comodule map and $\Delta_C \otimes \tilde{A}$ is a morphism in $\mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$. Dually to Definition 3.2 one considers

Definition 3.3. An admissible morphism $(f, g) : (A, C)_\psi \to (\tilde{A}, \tilde{C})_{\tilde{\psi}}$ of entwining structures is said to be:

(1) *cointegrable* if there exists $\mathfrak{z} \in \mathrm{Hom}^{\tilde{C}}(\tilde{C}, (\tilde{A} \otimes C) \otimes_A \tilde{A})$ such that the following diagrams commute

$$\begin{tikzcd}
\tilde{C} \arrow[r, "\mathfrak{z}"] \arrow[d, "\Delta_{\tilde{C}}"] & (\tilde{A} \otimes C) \otimes_A \tilde{A} \arrow[r, "(\tilde{A}\otimes\Delta_C)\otimes_A\tilde{A}"] & (\tilde{A} \otimes C \otimes C) \otimes_A \tilde{A} \arrow[d, "(\tilde{A}\otimes g\otimes C)\otimes_A\tilde{A}"] \\
\tilde{C} \otimes \tilde{C} \arrow[r, "\tilde{C}\otimes\mathfrak{z}"] & \tilde{C} \otimes (\tilde{A} \otimes C) \otimes_A \tilde{A} \arrow[r, "\tilde{\psi}\otimes\tilde{C}\otimes\tilde{A}"] & (\tilde{A} \otimes \tilde{C} \otimes C) \otimes_A \tilde{A}
\end{tikzcd} \tag{6}$$

$$\begin{tikzcd}
\tilde{C} \otimes \tilde{A} \arrow[r, "\tilde{\psi}"] \arrow[d, equal] & \tilde{A} \otimes \tilde{C} \arrow[r, "\tilde{A}\otimes\mathfrak{z}"] & (\tilde{A} \otimes \tilde{A} \otimes C) \otimes_A \tilde{A} \arrow[d, "(\mu_{\tilde{A}}\otimes C)\otimes_A\tilde{A}"] \\
\tilde{C} \otimes \tilde{A} \arrow[r, "\mathfrak{z}\otimes\tilde{A}"] & (\tilde{A} \otimes C) \otimes_A \tilde{A} \otimes \tilde{A} \arrow[r, "(\tilde{A}\otimes C)_A\otimes\mu_{\tilde{A}}"] & (\tilde{A} \otimes C) \otimes_A \tilde{A}.
\end{tikzcd} \tag{7}$$

The right $\tilde{C}$-comodule structure of $(\tilde{A} \otimes C) \otimes_A \tilde{A}$ is as in Example 2.6(2), explicitly $\rho^{(\tilde{A}\otimes C)\otimes_A\tilde{A}} : \tilde{a} \otimes c \otimes \tilde{a}' \mapsto \tilde{a} \otimes c_{(1)} \otimes \tilde{a}'_\alpha \otimes g(c_{(2)})^\alpha$.

(2) *totally cointegrable*, if there exists $\mathfrak{z} \in \mathrm{Hom}^{\tilde{C}}(\tilde{C}, (\tilde{A} \otimes C) \otimes_A \tilde{A})$ making it a cointegrable morphism and such that the following diagram

$$\begin{tikzcd}
\tilde{C} \arrow[r, "\mathfrak{z}"] \arrow[d, "1_{\tilde{A}}\circ\epsilon_{\tilde{C}}"] & (\tilde{A} \otimes C) \otimes_A \tilde{A} \arrow[d, "(\tilde{A}\otimes\epsilon_C)\otimes_A\tilde{A}"] \\
\tilde{A} & \tilde{A} \otimes_A \tilde{A} \arrow[l, "\mu_{\tilde{A}A}"']
\end{tikzcd} \tag{8}$$

commutes. Here $\mu_{\tilde{A},A} : \tilde{A} \otimes_A \tilde{A} \to \tilde{A}$ is the natural map induced by $\mu_{\tilde{A}}$.

The right actions of A on the k-modules involved in the above definition are as follows. For any $a \in A$, $\tilde{a}, \tilde{a}' \in \tilde{A}$, $c, c' \in C$, $\tilde{c} \in \tilde{C}$: $(\tilde{a} \otimes c) \cdot a = \tilde{a} f(a_\alpha) \otimes c^\alpha$, $(\tilde{a} \otimes c \otimes c') = \tilde{a} f(a_{\alpha\beta}) \otimes c^\beta \otimes c'^\alpha$, $(\tilde{a} \otimes \tilde{c} \otimes c) \cdot a = \tilde{a} f(a_\alpha)_\beta \otimes \tilde{c}^\beta \otimes c^\alpha$, $(\tilde{c} \otimes \tilde{a} \otimes c) \cdot a = \tilde{c} \otimes \tilde{a} f(a_\alpha) \otimes c^\alpha$, $(\tilde{a} \otimes \tilde{a}' \otimes c) \cdot a = \tilde{a} \otimes \tilde{a}' f(a_\alpha) \otimes c^\alpha$. Using properties of entwining structures and the fact that (f, g) is a morphism of entwining structures one can easily convince oneself that all the maps featuring in Definition 3.3 are well-defined.

With these definitions at hand we can now state the main result of this section.

Theorem 3.4. *Let $(f, g) : (A, C)_\psi \to (\tilde{A}, \tilde{C})_{\tilde{\psi}}$ be an admissible morphism of entwining structures.*

(1) If for all $M \in \mathbf{M}_A^C(\psi)$, $(M \otimes_A \tilde{A}) \Box_{\tilde{C}} C \subseteq \mathrm{coker}(\mathrm{eq}_{M_A\tilde{A}} \Box_{\tilde{C}} C)$, then the functor $- \otimes_A \tilde{A} : \mathbf{M}_A^C(\psi) \to \mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$ is separable if and only if (f, g) is totally integrable.

(2) If for all $\tilde{M} \in \mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi})$, $\ker(\mathrm{eq}^{\tilde{M}^{\tilde{C}}C} \otimes_A \tilde{A}) \subseteq (\tilde{M} \Box_{\tilde{C}} C) \otimes_A \tilde{A}$, then the functor $-\Box_{\tilde{C}} C : \mathbf{M}_{\tilde{A}}^{\tilde{C}}(\tilde{\psi}) \to \mathbf{M}_A^C(\psi)$ is separable if and only if (f, g) is totally cointegrable.

Proof. (1) Let (f, g) be totally integrable and assume that λ is as in Definition 3.2. For all $M \in \mathbf{M}_A^C(\psi)$ define

$$\tilde{\nu}_M : (M \otimes \tilde{A}) \Box_{\tilde{C}} C \to M, \qquad \sum_i m_i \otimes \tilde{a}_i \otimes c_i \mapsto \sum_i m_{i(0)} \cdot \lambda(m_{i(1)} \otimes \tilde{a}_i \otimes c_i).$$

Notice that the map $\tilde{\nu}_M$ is well-defined since the fact that (f, g) is admissible implies that for any $x \in (M \otimes \tilde{A}) \Box_{\tilde{C}} C$, one has $(\rho^M \otimes \tilde{A} \otimes C)(x) \in (M \otimes (C \otimes \tilde{A})) \Box_{\tilde{C}} C) =$

$M\otimes((C\otimes\tilde{A})\Box_{\tilde{C}}C)$. Take any $x=\sum_i m_i\cdot a_i\otimes\tilde{a}_i\otimes c_i\in(M\otimes\tilde{A})\Box_{\tilde{C}}C$. Then

$$\begin{aligned}
\tilde{\nu}_M(x) &= \sum_i (m_i\cdot a_i)_{(0)}\cdot\lambda((m_i\cdot a_i)_{(1)}\otimes\tilde{a}_i\otimes c_i)\\
&= \sum_i m_{i(0)}\cdot a_{i\alpha}\lambda({m_{i(1)}}^{\alpha}\otimes\tilde{a}_i\otimes c_i) \qquad (M\in\mathbf{M}_A^C(\psi))\\
&= \sum_i m_{i(0)}\cdot\lambda(m_{i(1)}\otimes f(a_i)\tilde{a}_i\otimes c_i) \qquad (\text{by } (3))\\
&= \tilde{\nu}_M(\sum_i m_i\otimes f(a_i)\tilde{a}_i\otimes c_i).
\end{aligned}$$

The above calculation means that $\mathrm{Im}(\mathrm{eq}_{M_A\tilde{A}}\Box_{\tilde{C}}C)\subseteq\ker\tilde{\nu}_M$, and together with the assumption that $-\Box_{\tilde{C}}C$ preserves the cokernel of the action equalising map $\mathrm{eq}_{M_A\tilde{A}}$ imply that one can define the map $\nu_M:(M\otimes_A\tilde{A})\Box_{\tilde{C}}C\to M$ by the diagram

$$\begin{array}{ccccccc}
(M\otimes_A\tilde{A})\Box_{\tilde{C}}C & \longrightarrow & \mathrm{coker}(\mathrm{eq}_{M_A\tilde{A}}\Box_{\tilde{C}}C) & \longrightarrow & ((M\otimes\tilde{A})\Box_{\tilde{C}}C)/\ker\tilde{\nu}_M & \longrightarrow & M\\
 & & & & \uparrow & & \|\\
 & & & & (M\otimes\tilde{A})\Box_{\tilde{C}}C & \xrightarrow{\tilde{\nu}_M} & M
\end{array}$$

Slightly abusing the notation we will still write $\nu_M:\sum_i m_i\otimes\tilde{a}_i\otimes c_i\mapsto\sum_i m_{i(0)}\cdot\lambda(m_{i(1)}\otimes\tilde{a}_i\otimes c_i)$.

To show that ν_M is a right A-module map, take any $a\in A$ and $x=\sum_i m_i\otimes\tilde{a}_i\otimes c_i\in(M\otimes_A\tilde{A})\Box_{\tilde{C}}C$ and compute

$$\begin{aligned}
\nu_M(x\cdot a) &= \nu_M(\sum_i m_i\otimes\tilde{a}_i f(a_\alpha)\otimes c_i^{\alpha})\\
&= \sum_i m_{i(0)}\cdot\lambda(m_{i(1)}\otimes\tilde{a}_i f(a_\alpha)\otimes c_i^{\alpha})\\
&= \sum_i m_{i(0)}\cdot\lambda(m_{i(1)}\otimes\tilde{a}_i\otimes c_i)a \qquad (f\in\mathrm{Hom}_A((C\otimes\tilde{A})\Box_{\tilde{C}}C,A))\\
&= \nu_M(x)\cdot a.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
\nu_M(x_{(0)})\otimes x_{(1)} &= \sum_i m_{i(0)}\cdot\lambda(m_{i(1)}\otimes\tilde{a}_i\otimes c_{i(1)})\otimes c_{i(2)}\\
&= \sum_i m_{i(0)}\cdot\lambda(m_{i(2)}\otimes\tilde{a}_i\otimes c_i)_\alpha\otimes {m_{i(1)}}^{\alpha} \qquad (\text{by } (4))\\
&= \sum_i \rho^M(m_{i(0)}\cdot\lambda(m_{i(1)}\otimes\tilde{a}_i\otimes c_i)) \qquad (M\in\mathbf{M}_A^C(\psi))\\
&= \rho^M\circ\nu_M(x),
\end{aligned}$$

which proves that ν_M is a right C-comodule map. Using (5) one easily finds that the adjunction Φ_M is splitted by ν_M. It remains to be shown that ν_M is natural in

$\mathbf{M}_A^C(\psi)$. Take any $M, N \in \mathbf{M}_A^C(\psi)$ and $\phi \in \mathrm{Hom}_A^C(M, N)$. Then

$$\begin{aligned}\nu_N(\sum_i \phi(m_i) \otimes \tilde{a}_i \otimes c_i) &= \sum_i \phi(m_i)_{(0)} \cdot \lambda(\phi(m_i)_{(1)} \otimes \tilde{a}_i \otimes c_i) \\ &= \sum_i \phi(m_{i(0)}) \cdot \lambda(m_{i(1)} \otimes \tilde{a}_i \otimes c_i) \\ &= \sum_i \phi(m_{i(0)} \cdot \lambda(m_{i(1)} \otimes \tilde{a}_i \otimes c_i)) \\ &= \phi \circ \nu_M(x),\end{aligned}$$

where we used that ϕ is a right C-comodule and right A-module map to derive the second and the third equalities respectively. This completes the proof that the functor $-\Box_{\tilde{C}} C$ is separable.

Conversely, assume that $-\Box_{\tilde{C}} C$ is separable and let ν_M be the corresponding splitting of Φ_M. Define

$$\lambda : (C \otimes \tilde{A})\Box_{\tilde{C}} C \to A, \qquad \lambda = (A \otimes \epsilon_C) \circ \nu_{A\otimes C}(1_A \otimes (C \otimes \tilde{A})\Box_{\tilde{C}} C).$$

Since $\nu_{A\otimes C}$ is a right A-linear map, so is λ. We first show that ν_M can be expressed in terms of λ. For any $M \in \mathbf{M}_A$ and $m \in M$ consider a morphism $\ell_m : A\otimes C \to M\otimes C$ in $\mathbf{M}_A^C(\psi)$ given by $a \otimes c \mapsto m \cdot a \otimes c$. Since the splitting of the adjunction Φ is natural in $\mathbf{M}_A^C(\psi)$ we have

$$\ell_m \circ \nu_{A\otimes C} = \nu_{M\otimes C} \circ ((\ell_m \otimes_A \tilde{A})\Box_{\tilde{C}} C). \tag{9}$$

In particular, choosing $M = A$ one easily finds that (9) implies that $\nu_{A\otimes C}$ is a left A-module map. Now, if $M \in \mathbf{M}_A^C(\psi)$ one can take the morphism $\rho^M \in \mathrm{Hom}_A^C(M, M\otimes C)$, and thus using the naturality of ν, obtain $\rho^M \circ \nu_M = \nu_{M\otimes C} \circ (\rho^M \otimes C)$. In view of (9) this reads for all $\sum_i m_i \otimes \tilde{a}_i \otimes c_i \in (M \otimes \tilde{A})\Box_{\tilde{C}} C$ projected down to $(M \otimes_A \tilde{A})\Box_{\tilde{C}} C$

$$\rho^M \circ \nu_M(\sum_i m_i \otimes \tilde{a}_i \otimes c_i) = \sum_i \ell_{m_{i(0)}} \circ \nu_{A\otimes C}(1_A \otimes m_{i(1)} \otimes \tilde{a}_i \otimes c_i).$$

Applying $M \otimes \epsilon_C$ to this last equality and using assumption that (f, g) is admissible one obtains

$$\nu_M(\sum_i m_i \otimes \tilde{a}_i \otimes c_i) = \sum_i m_{i(0)} \cdot \lambda(m_{i(1)} \otimes \tilde{a}_i \otimes c_i).$$

In particular, the choice $M = A\otimes C$ gives for all $a \in A$, $\sum_i c_i \otimes \tilde{a}_i \otimes c'_i \in (C \otimes \tilde{A})\Box_{\tilde{C}} C$

$$\nu_{A\otimes C}(\sum_i a \otimes c_i \otimes \tilde{a}_i \otimes c'_i) = a \sum_i \lambda(c_{i(2)} \otimes \tilde{a}_i \otimes c'_i)_\alpha \otimes {c_{i(1)}}^\alpha. \tag{10}$$

We are now ready to show that λ satisfies all the conditions of Definition 3.2. Take any $x = \sum_i c_i \otimes \tilde{a}_i \otimes c'_i \in (C \otimes \tilde{A})\Box_{\tilde{C}} C$, then

$$\begin{aligned}\sum_i \lambda(c_i \otimes \tilde{a}_i \otimes c'_{i(1)}) \otimes c'_{i(2)} &= \sum_i (A \otimes \epsilon_C) \circ \nu_{A\otimes C}(1_A \otimes c_i \otimes \tilde{a}_i \otimes c'_{i(1)}) \otimes c'_{i(2)} \\ &= (A \otimes \epsilon_C \otimes C) \circ (A \otimes \Delta_C) \circ \nu_{A\otimes C}(1_A \otimes x) \\ &= \nu_{A\otimes C}(1_A \otimes x) \\ &= \sum_i \lambda(c_{i(1)} \otimes \tilde{a}_i \otimes c'_i)_\alpha \otimes {c_{i(2)}}^\alpha \qquad \text{(by (10))},\end{aligned}$$

where we used that $\nu_{A\otimes C}$ is a right C-comodule map to derive the second equality. This proves that λ satisfies (4). Furthermore, for all $\sum_i c_i\otimes a_i\otimes \tilde{a}_i\otimes c_i'\in (C\otimes A\otimes \tilde{A})\Box_{\tilde{C}}C$ we have

$$\begin{aligned}\lambda(\sum_i c_i\otimes f(a_i)\tilde{a}_i\otimes c_i') &= (A\otimes\epsilon_C)\circ\nu_{A\otimes C}(\sum_i 1_A\otimes c_i\otimes f(a_i)\tilde{a}_i\otimes c_i')\\ &= (A\otimes\epsilon_C)\circ\nu_{A\otimes C}(\sum_i (1_A\otimes c_i)\cdot a_i\otimes \tilde{a}_i\otimes c_i')\\ &= (A\otimes\epsilon_C)\circ\nu_{A\otimes C}(\sum_i a_{i\alpha}\otimes c_i^\alpha\otimes \tilde{a}_i\otimes c_i')\\ &= \sum_i a_{i\alpha}\lambda(c_i^\alpha\otimes\tilde{a}_i\otimes c_i'),\end{aligned}$$

where we used the properties of the domain of $\nu_{A\otimes C}$ and the assumption that $-\Box_{\tilde{C}}C$ preserves cokernel of $\mathrm{eq}_{M_A\tilde{A}}$ to derive the second equality. This proves that λ satisfies (3). Finally, for all $c\in C$, $\Phi_{A\otimes C}(1_A\otimes c) = 1_A\otimes c_{(1)}\otimes 1_{\tilde{A}}\otimes c_{(2)}$. Since $\nu_{A\otimes C}$ splits $\Phi_{A\otimes C}$ we have $1\otimes c = \nu_{A\otimes C}(1_A\otimes c_{(1)}\otimes 1_{\tilde{A}}\otimes c_{(2)})$. Applying $A\otimes\epsilon_C$ to this equality one immediately deduces that λ satisfies (5). Therefore the morphism (f,g) is totally integrable. This completes the proof of the first statement of the theorem.

(2) Given $\mathfrak{z}$ as in Definition 3.3 define for all $\tilde{M}\in\mathbf{M}_A^C(\tilde{\psi})$, $\nu_{\tilde{M}}:\tilde{M}\to(\tilde{M}\Box_{\tilde{C}}C)\otimes_A \tilde{A}$, $\nu_{\tilde{M}} = (\rho_{\tilde{M}}\otimes C\otimes_A\tilde{A})\circ(\tilde{M}\otimes\mathfrak{z})\circ\rho^{\tilde{M}}$. The proof that $\nu_{\tilde{M}}$ is the required cosplitting is dual to the proof of the corresponding part of assertion (1). Conversely, given a cosplitting $\nu_{\tilde{M}}$ define $\zeta = (\epsilon_{\tilde{C}}\otimes\tilde{A}\otimes C\otimes_A\tilde{A})\circ\nu_{\tilde{C}\otimes\tilde{A}}\circ(\tilde{C}\otimes 1_{\tilde{A}})$. □

Notice that the assumption of Theorem 3.4(1) is satisfied if ${}^{\tilde{C}}C$ is coflat. Dually, the assumption of Theorem 3.4(2) is satisfied if ${}_A\tilde{A}$ is flat. The remainder of the paper is devoted to the analysis of special cases of Theorem 3.4.

4. Separable coalgebra-Galois extensions

The following notion was introduced in [1]. It generalises the notion of an H-integral for a Doi-Hopf datum [6, Definition 2.1].

Definition 4.1. Let $(A,C)_\psi$ be an entwining structure. An *integral* in $(A,C)_\psi$ is an element $\mathfrak{z} = \sum_i a_i\otimes c_i\in A\otimes C$ such that for all $a\in A$, $a\cdot\mathfrak{z} = \mathfrak{z}\cdot a$. Explicitly, we require $\sum_i aa_i\otimes c_i = \sum_i a_i\psi(c_i\otimes a)$. An integral $\mathfrak{z} = \sum_i a_i\otimes c_i$ is said to be *normalised* if $\sum_i a_i\epsilon_C(c_i) = 1$.

Example 4.2. Let A be a Hopf algebra and $B\subset A$ be a left A-comodule subalgebra, i.e., a subalgebra of A such that $\Delta_A(B)\subset A\otimes B$. Consider the coalgebra C/B^+A. C is a right A-module in the natural way and there is an entwining structure $(A,C)_\psi$ with $\psi: c\otimes a\mapsto a_{(1)}\otimes c\cdot a_{(2)}$. Let $\Lambda\in C$ be such that for all $a\in A$, $\Lambda\cdot a = \epsilon_A(a)\Lambda$ and $\epsilon_C(\Lambda) = 1$. Then $\mathfrak{z} = 1\otimes\Lambda$ is an integral in $(A,C)_\psi$.

Proof. Clearly, $1_A\epsilon(\Lambda) = 1_A$. Take any $a\in A$, then $(1\otimes\Lambda)\cdot a = a_{(1)}\otimes\Lambda\cdot a_{(2)} = a\otimes\Lambda = a\cdot(1\otimes\Lambda)$. □

In [1] it has been shown that the existence of an integral in $(A,C)_\psi$ is closely related to the fact that the functor $-\otimes C:\mathbf{M}_A\to\mathbf{M}_A^C(\psi)$ of Example 2.7(1) is both left and right adjoint of the forgetful functor $\mathbf{M}_A^C(\psi)\to\mathbf{M}_A$. The following

theorem, which is an entwining structure version of [5, Theorem 2.14], shows that integrals are closely related to the separability of $-\otimes C$.

Theorem 4.3. *Let $(A,C)_\psi$ be an entwining structure. The functor $-\otimes C : \mathbf{M}_A \to \mathbf{M}_A^C(\psi)$ is separable if and only if there exists a normalised integral in $(A,C)_\psi$.*

Proof. Consider an admissible morphism $(A,\epsilon_C) : (A,C)_\psi \to (A,k)_{\rm twist}$. Then $-\otimes C = -\Box_k C : \mathbf{M}_A \to \mathbf{M}_A^C(\psi)$. Since ${}_AA$ is flat, Theorem 3.4(2) can be applied and thus $-\otimes C$ is separable if and only if (A,ϵ_C) is totally cointegrable, i.e. there exists $\mathfrak{z} \in \mathrm{Hom}(k,(A\otimes C)\otimes_A A) \cong A\otimes C$ such that conditions (6)–(8) are satisfied. In this case condition (6) is empty, while condition (7) means that $\mathfrak{z}$ is an integral in $(A,C)_\psi$. Finally, condition (8) states that $\mathfrak{z}$ is normalised. □

The existence of normalised integrals in the canonical entwining structure associated to a coalgebra-Galois extensions turns out to be equivalent to the separability of such an extension. First, recall from [13]

Definition 4.4. An extension of algebras $B \hookrightarrow A$ is *separable* if there exists $u \in A\otimes_B A$ such that for all $a\in A$, $au = ua$ and $\mu_{A,B}(u) = 1_A$, where $\mu_{A,B} : A\otimes_B A \to A$ is the natural map induced by μ_A. The element u is called a *separability idempotent.*

Proposition 4.5. *A coalgebra-Galois extension $A(B)^C$ is separable if and only if there exists a normalised integral in the canonical entwining structure.*

Proof. We first show that $\mathrm{can}^{-1} : A\otimes C \to A\otimes_B A$ is an (A,A)-bimodule map, where the (A,A)-bimodule structure on $A\otimes C$ is as in Definition 4.1. By construction, can^{-1} is a left A-module map. For all $\mathfrak{z} = \sum_i a_i\otimes c_i \in A\otimes C$, $a\in A$

$$\begin{aligned}
\mathrm{can}^{-1}(\mathfrak{z}\cdot a) &= \mathrm{can}^{-1}(\sum_i a_i a_\alpha \otimes c_i^\alpha) = \sum_i a_i \mathrm{can}^{-1}(a_\alpha\otimes c_i^\alpha)\\
&= \sum_i a_i \mathrm{can}^{-1}(\mathrm{can}(\mathrm{can}^{-1}(1_A\otimes c_i)a)) \qquad \text{(def. of canonical } \psi)\\
&= \sum_i a_i \mathrm{can}^{-1}(1_A\otimes c_i)a = \mathrm{can}^{-1}(\mathfrak{z})a.
\end{aligned}$$

Therefore $\mathfrak{z}$ is an integral in $(A,C)_\psi$ if and only if for all $a\in A$, $au = ua$, where $u = \mathrm{can}^{-1}(\mathfrak{z})$. Furthermore, directly from the definition of the canonical map can, one finds that $(A\otimes\epsilon_C)\circ \mathrm{can} = \mu_{A,B}$. Therefore $\mathfrak{z}$ is normalised if and only if $\mu_{A,B}(u) = 1_A$. □

Example 4.6. In the setting of Example 4.2, view A as a right C-comodule via $\rho^A = (A\otimes\pi)\circ\Delta_A$, where $\pi : A\to C = A/B^+A$ is the canonical surjection, and assume that $B = \{b\in B \mid \forall a\in A,\ \rho^A(ba) = b\rho^A(a)\}$ (for example, this holds if either ${}_BA$ or A_B is faithfully flat). Then $A(B)^C$ is a coalgebra-Galois extension, and if there is $\Lambda\in C$ such that for all $a\in A$, $\Lambda\cdot a = \epsilon_A(a)\Lambda$ and $\epsilon_C(\Lambda) = 1$, then $B\hookrightarrow A$ is separable.

The introduction of separable extensions in [13] was motivated by the Hochschild relative homological algebra [12]. In the case of a coalgebra-Galois extension the relationship between cohomology and separable extensions can be expressed in terms of integrals in the canonical entwining structure. Recall from [12] that if B is

a subalgebra of A then for every (A,A)-bimodule M the *relative Hochschild cohomology groups* $H^n(A,B,M)$ are defined as cohomology groups of the complex $(\bigoplus_{n=0}^{\infty} C^n(A,B,M),\delta)$, where $C^0(A,B,M)=\{m\in M \mid \forall b\in B,\ b\cdot m=m\cdot b\}$,

$$C^n(A,B,M) = {}_B\mathrm{Hom}_B(\underbrace{A\otimes_B A\otimes_B\cdots\otimes_B A}_{n\text{-times}},M), \qquad n>0,$$

and the coboundary $\delta : C^n(A,B,M)\to C^{n+1}(A,B,M)$ is given by

$$\begin{aligned}\delta(f)(a_1,\ldots,a_{n+1}) &= a_1\cdot f(a_2,\ldots a_{n+1})+\sum_{i=1}^{n}(-1)^i f(a_1,\ldots,a_ia_{i+1},\ldots,a_{n+1})\\ &\quad +(-1)^{n+1}f(a_1,\ldots,a_n)\cdot a_{n+1}.\end{aligned}$$

Corollary 4.7. *Let $A(B)^C$ be a coalgebra-Galois extension. Then a normalised integral in the associated canonical entwining structure exists if and only if for all (A,A)-bimodules M, $H^1(A,B,M)=0$.*

Proof. By an argument similar to [9, p. 76], one shows, that the first relative Hochschild cohomology group is trivial for all (A,A)-bimodules if and only if the extension $B\hookrightarrow A$ is separable. Then the assertion follows from Proposition 4.5. □

Corollary 4.8. *Let $A(B)^C$ be a coalgebra-Galois extension with a normalised integral in the canonical entwining structure. Then any (A,A)-bimodule which is semisimple as a (B,B)-bimodule is semisimple as an (A,A)-bimodule.*

Proof. By Corollary 4.7, for all (A,A)-bimodules M, $H^1(A,B,M)=0$. Then [12, Theorem 1] implies the assertion. □

Dually one can consider

Definition 4.9. Let (A,C,ψ) be an entwining structure. A k-module map $\mathfrak{y}$: $C\otimes A\to k$, such that for all $a\in A$, $c\in C$, $c_{(1)}\mathfrak{y}(c_{(2)}\otimes a)=\mathfrak{y}(c_{(1)}\otimes a_\alpha)c_{(2)}{}^\alpha$ is called a *cointegral* in $(A,C)_\psi$. A cointegral $\mathfrak{y}$ is said to be *normalised* if $\mathfrak{y}\circ(C\otimes 1_A)=\epsilon_C$.

Example 4.10. Let C be a Hopf algebra and let A be a right C-comodule algebra. Then $(A,C)_\psi$ is an entwining structure with $\psi : c\otimes a\mapsto a_{(0)}\otimes ca_{(1)}$. Let $\kappa\in A^*$ be such that $\kappa(1_A)=1$ and for all $a\in A$, $1_C\kappa(a)=\kappa(a_{(0)})a_{(1)}$. Then $\mathfrak{y}=\epsilon_C\otimes\kappa$ is a normalised cointegral in $(A,C)_\psi$.

Theorem 4.11. *Let $(A,C)_\psi$ be an entwining structure. The functor $-\otimes A:\mathbf{M}^C\to\mathbf{M}_A^C(\psi)$ is separable if and only if there exists a normalised cointegral in $(A,C)_\psi$.*

Proof. Consider the morphism $(1_A,C):(k,C)_{\mathrm{twist}}\to(A,C)_\psi$ and apply Theorem 3.4(1). □

Definition 4.12. A coextension of coalgebras $C\twoheadrightarrow B$ is said to be a *separable coextension* if there exists a k-module map $\upsilon : C\Box_B C\to k$ such that $(C\otimes\upsilon)\circ(\Delta_C\otimes C)=(\upsilon\otimes C)\circ(C\otimes\Delta_C)$ on $C\Box_B C$, and $\upsilon\circ\Delta_C=\epsilon_C$.

Proposition 4.13. *An algebra-Galois coextension $C(B)_A$ is separable if and only if there exists a normalised cointegral in the associated canonical entwining structure.*

Example 4.14. Let C be a Hopf algebra and $A \subset C$ a right comodule subalgebra of C, i.e., $\Delta_C(A) \subset A \otimes C$, so that we are in the setting of Example 4.10. Consider the coalgebra $B = C/CA^+$, and assume that $A = \{a \in C \mid \pi(a_{(1)}) \otimes a_{(2)} = \pi(1_C) \otimes a\}$, where $\pi : C \to B$ is the canonical surjection (this assumption is satisfied if either ${}_AC$ or C_A is faithfully flat). Then $C \twoheadrightarrow B$ is an A-Galois coextension and if there exists $\kappa \in A^*$ such that for all $a \in A$, $\kappa(a_{(1)})a_{(2)} = \kappa(a)\epsilon_C$ and $\kappa(1_A) = 1$, then this coextension is separable.

When $k = \mathbf{C}$, a rich source of separable coalgebra-Galois extensions is provided by quantum homogeneous spaces of compact quantum groups [21]. In this case we are in the setting of Example 4.14, with C a compact quantum group and A a right C-homogeneous quantum space. In many cases C is a faithfully flat right or left A-module (see [16] for examples). The map κ is the Haar measure on C restricted to A. Perhaps the simplest example of this situation is when C is the quantum $SU(2)$ group and A is any of the quantum 2-spheres of Podleś [19].

5. Split coalgebra-Galois extensions

The following definition is a slightly modified version of [1, Definition 4.1]; both definitions describe the same object if a coalgebra C is a finitely-generated projective k-module.

Definition 5.1. Let $(A,C)_\psi$ be an entwining structure. Any $\gamma \in \mathrm{Hom}(C \otimes C, A)$ such that the following diagrams

$$\begin{tikzcd} C \otimes C \arrow[r, "C\otimes\Delta_C"] \arrow[d, "\Delta_C\otimes C"'] & C \otimes C \otimes C \arrow[r, "\gamma\otimes C"] & A \otimes C \arrow[d, equal] \\ C \otimes C \otimes C \arrow[r, "C\otimes\gamma"] & C \otimes A \arrow[r, "\psi"] & A \otimes C \end{tikzcd} \tag{11}$$

$$\begin{tikzcd} C \otimes C \otimes A \arrow[r, "\gamma\otimes A"] \arrow[d, "C\otimes\psi"'] & A \otimes A \arrow[r, "\mu_A"] & A \\ C \otimes A \otimes C \arrow[r, "\psi\otimes C"] & A \otimes C \otimes C \arrow[r, "A\otimes\gamma"] & A \otimes A \arrow[u, "\mu_A"'] \end{tikzcd} \tag{12}$$

commute is called an *integral map* in $(A,C)_\psi$. An integral map γ is said to be *normalised*, if for all $c \in C$, $\gamma(c_{(1)} \otimes c_{(2)}) = \epsilon_C(c)1_A$.

The following theorem is an entwining structure version of [5, Theorem 2.3].

Theorem 5.2. *The forgetful functor $\mathbf{M}_A^C(\psi) \to \mathbf{M}_A$ is separable if and only if there exists a normalised integral map in $(A,C)_\psi$.*

Proof. Consider an admissible morphism $(A,\epsilon_C) : (A,C)_\psi \to (A,k)_{\mathrm{twist}}$. Then $-\otimes_A A : \mathbf{M}_A^C(\psi) \to \mathbf{M}_A$ is the forgetful functor. In this case $(C\otimes A)\Box_k C = C\otimes A\otimes C$, and for all $M \in \mathbf{M}_A$, $\mathrm{eq}_{M_A A} = M$ so that the assumption of Theorem 3.4(1) holds. Therefore the forgetful functor is separable if and only if (A,ϵ_C) is totally integrable, i.e., iff there exists $\lambda \in \mathrm{Hom}_A(C\otimes A\otimes C, A)$ satisfying all the conditions of Definition 3.2. Assume that such a λ exists and define $\gamma = \lambda \circ (C \otimes 1_A \otimes C) : C \otimes C \to A$. Then for all $a \in A$, $c, c' \in C$ we have

$$\begin{aligned} a_{\alpha\beta}\gamma(c^\beta \otimes c'^\alpha) &= a_{\alpha\beta}\lambda(c^\beta \otimes 1_A \otimes c'^\alpha) = \lambda(c \otimes a_\alpha \otimes c'^\alpha) \qquad \text{(by (3))} \\ &= \lambda(c \otimes 1_A \otimes c')a = \gamma(c \otimes c')a, \end{aligned}$$

where we used that λ is a right A-module map to derive the penultimate equality. Hence the diagram (11) commutes. Also, (4) implies that the diagram (12) commutes, while the normalisation of γ follows immediately from (5). Thus we conclude that γ is a normalised integral map as required.

Conversely, assume that γ is a normalised integral map and define $\lambda : C\otimes A\otimes C \to A$, $c\otimes a\otimes c' \mapsto a_\alpha\gamma(c^\alpha\otimes c')$. For all $a, a' \in A$, $c, c' \in C$ we have

$$a_\alpha\lambda(c^\alpha\otimes a'\otimes c') = a_\alpha a'_\beta\gamma(c^{\alpha\beta}\otimes c') = (aa')_\alpha\gamma(c^\alpha\otimes c') = \lambda(c\otimes aa'\otimes c'),$$

where (1) was used to obtain the third equality. This proves that the diagram (3) commutes. Furthermore

$$\begin{aligned}
\lambda(c_{(2)}\otimes a\otimes c')_\alpha\otimes {c_{(1)}}^\alpha &= (a_\delta\gamma({c_{(2)}}^\delta\otimes c'))_\alpha\otimes {c_{(1)}}^\alpha \\
&= a_{\delta\alpha}\gamma({c_{(2)}}^\delta\otimes c')_\beta\otimes {c_{(1)}}^{\alpha\beta} && \text{(by (1))}\\
&= a_\alpha\gamma({c^\alpha}_{(2)}\otimes c')_\beta\otimes {{c^\alpha}_{(1)}}^\beta && \text{(by (2))}\\
&= a_\alpha\gamma(c^\alpha\otimes c'_{(1)})_\beta\otimes c'_{(2)} && \text{(by (12))}\\
&= \lambda(c\otimes a\otimes c'_{(1)})\otimes c'_{(2)}.
\end{aligned}$$

This proves that diagram (4) commutes. Also,

$$\begin{aligned}
\lambda(c\otimes aa'_\alpha\otimes c'^\alpha) &= (aa'_\alpha)_\beta\gamma(c^\beta\otimes c'^\alpha) = a_\beta a'_{\alpha\delta}\gamma(c^{\beta\delta}\otimes c'^\alpha) && \text{(by (1))}\\
&= a_\beta\gamma(c^\beta\otimes c')a' = \lambda(c\otimes a\otimes a') && \text{(by (11))}.
\end{aligned}$$

Therefore λ is a right A-module map, and, consequently the morphism (A,ϵ_C) is integrable. The fact that it is totally integrable follows immediately from the normalisation of γ. □

Example 5.3. Let $(A,C)_\psi$ be the canonical entwining structure associated to a pointed algebra-Galois coextension $C(k)_A$ of k. Then the forgetful functor $\mathbf{M}_A^C(\psi)\to \mathbf{M}_A$ is separable.

Proof. Since $B = k$, $C\Box_B C = C\otimes C$, and we define $\gamma = (\epsilon_C\otimes A)\circ \mathrm{cocan}^{-1} : C\otimes C\to A$. We show that γ is a normalised integral map. First notice that since cocan^{-1} is a left C-comodule map, one has $\mathrm{cocan}^{-1} = (C\otimes\gamma)\circ(\Delta_C\otimes C)$. Applying the definition of the canonical entwining map in Example 2.3 to cocan^{-1} one immediately obtains $\psi\circ\mathrm{cocan}^{-1} = (\gamma\otimes C)\circ(C\otimes\Delta_C)$, i.e. $\psi\circ(C\otimes\gamma)\circ(\Delta_C\otimes C) = (\gamma\otimes C)\circ(C\otimes\Delta_C)$. Thus we conclude that γ satisfies condition (11).

Let $\kappa : A\to k$ be the algebra map making $C(k)_A$ a pointed algebra-Galois coextension. One easily finds that $\rho_C = (\kappa\otimes C)\circ\psi$ and $C\otimes\kappa = (C\otimes\epsilon_C)\circ\mathrm{cocan}$. The map γ is the *cotranslation map*, so, as explained in [3, Theorem 3.5], it has the following properties

$$\mu_A\circ(\gamma\otimes A) = \gamma\circ(C\otimes\rho_C), \tag{13}$$

$$\mu_A\circ(\gamma\otimes\gamma)\circ(C\otimes\Delta_C\otimes C) = \gamma\circ(C\otimes\epsilon_C\otimes C). \tag{14}$$

Using all these properties we obtain

$$
\begin{aligned}
\mu_A \circ (\gamma \otimes A) &= \gamma \circ (C \otimes \rho_C) && \text{(by (13))} \\
&= \gamma \circ (C \otimes \kappa \otimes C) \circ (C \otimes \psi) \\
&= \gamma \circ (C \otimes \epsilon_C \otimes C) \circ (\mathrm{cocan} \otimes C) \circ (C \otimes \psi) \\
&= \mu_A \circ (\gamma \otimes \gamma) \circ (C \otimes \Delta_C \otimes C) \circ (\mathrm{cocan} \otimes C) \circ (C \otimes \psi) && \text{(by (14))} \\
&= \mu_A \circ (A \otimes \gamma) \circ (\psi \otimes C) \circ (C \otimes \psi) && \text{(def. of } \psi).
\end{aligned}
$$

This proves that γ is an integral map. Finally, γ is normalised by the normalisation property of the cotranslation map (cf. [3, Theorem 3.5]). □

As explained in [5] the separability of the forgetful functor implies various Maschke-type theorems. Thus, similarly as in [1] we have

Corollary 5.4. *If there is a normalised integral map in $(A, C)_\psi$, then*

(1) Every object in $\mathbf{M}_A^C(\psi)$ which is semisimple as an object in $\mathbf{M}_A$ is semisimple as an object in $\mathbf{M}_A^C(\psi)$.

(2) Every object in $\mathbf{M}_A^C(\psi)$ which is projective (resp. injective) as a right A-module is a projective (resp. injective) object in $\mathbf{M}_A^C(\psi)$.

(3) $M \in \mathbf{M}_A^C(\psi)$ is projective as a right A-module if and only if there exists $V \in \mathbf{M}^C$ such that M is a direct summand of $V \otimes A$ in $\mathbf{M}_A^C(\psi)$ ($V \otimes A$ is an entwined module by Example 2.7(2)).

In the case of a coalgebra-Galois extension, the existence of normalised integral maps in the canonical entwining structure is closely related to the coalgebra-Galois extension being a split extension. Recall from [18][14]

Definition 5.5. An extension of algebras $B \hookrightarrow A$ is called a *split extension* if there exists a unital (B, B)-bimodule map $E : A \to B$. The map E is called a *conditional expectation.*

Proposition 5.6. *A coalgebra-Galois extension $A(B)^C$ is a split extension if and only if there exists $\phi \in \mathrm{Hom}(C, A)$ such that*

(i) $\forall c \in C, \qquad \psi(c_{(1)} \otimes \phi(c_{(2)})) = \phi(c)\rho^A(1_A)$,

(ii) $\sum_i a^i \phi(c_i) = 1_A$, where $\sum_i a^i \otimes c_i = \rho^A(1_A)$.

(iii) $\forall b \in B,\ c \in C, \qquad b_\alpha \phi(c^\alpha) = \phi(c)b$.

Proof. As explained in the proof of [2, Proposition 4.4], given a unital (B, B)-bimodule map $E : A \to B$ there exists $\phi \in \mathrm{Hom}(C, A)$ satisfying conditions (i)–(iii). Explicitly, $\phi = (A \otimes_B E) \circ \mathrm{can}^{-1} \circ (1_A \otimes C)$. Conversely, given $\phi \in \mathrm{Hom}(C, A)$ satisfying (i), [2, Theorem 4.3] implies that $E : A \to B$, $a \mapsto a_{(0)}\phi(a_{(1)})$ is a left B-module map. Clearly, condition (ii) implies E is unital. Furthermore, for all $a \in A$, $b \in B$

$$E(ab) = (ab)_{(0)}\phi((ab)_{(1)}) = a_{(0)}b_\alpha\phi({a_{(1)}}^\alpha) = a_{(0)}\phi(a_{(1)})b = E(a)b,$$

where we used that $A \in \mathbf{M}_A^C(\psi)$ and the assumption (iii) to derive the second and third equalities respectively. This completes the proof. □

Let $(A, C)_\psi$ be an entwining structure and assume that $A \in \mathbf{M}_A^C(\psi)$. Define B as in Example 2.2. Then one can consider a covariant functor $(-)_0 : \mathbf{M}_A^C(\psi) \to \mathbf{M}_B$

$$M \mapsto M_0 := \{m \in M \mid \forall a \in A,\ \ \rho^M(m \cdot a) = m\rho^A(a)\}.$$

Notice, in particular, that $B = A_0$. As explained in [2] the functor $(-)_0$ is the right adjoint of the functor $- \otimes_B A : \mathbf{M}_B \to \mathbf{M}_A^C(\psi)$.

Corollary 5.7. *If a coalgebra-Galois extension $A(B)^C$ is a split extension then ${}_BA$ is a faithfully flat module. Consequently, the functors $- \otimes_B A : \mathbf{M}_B \to \mathbf{M}_A^C(\psi)$ and $(-)_0 : \mathbf{M}_A^C(\psi) \to \mathbf{M}_B$ are inverse equivalences.*

Proof. The first assertion follows from [2, Proposition 4.4], while the second is the consequence of [2, Corollary 3.11]. □

Proposition 5.8. *Let $(A, C)_\psi$ be the canonical entwining structure associated to a coalgebra-Galois extension $A(B)^C$. If there is a normalised integral map in $(A, C)_\psi$ then $B \hookrightarrow A$ is a split extension.*

Proof. Let $\gamma : C \otimes C \to A$ be a normalised integral map in $(A, C)_\psi$, and take $\phi : C \to A$, $c \mapsto \sum_i a^i_\alpha \gamma(c^\alpha \otimes c_i)$, where $\sum_i a^i \otimes c_i = \rho^A(1_A)$. Notice that the fact that A is an $(A, C)_\psi$ module implies that for all $a \in A$, $\rho^A(a) = \sum_i a^i a_\alpha \otimes c_i^\alpha$. Furthermore, since ρ^A is a coaction we have

$$\sum_{i,j} a^j a^i_\alpha \otimes c_j^\alpha \otimes c_i = \sum_i a^i \otimes c_{i(1)} \otimes c_{i(2)}. \tag{15}$$

We now show that ϕ satisfies all the conditions of Proposition 5.6. For all $c \in C$

$$\begin{aligned}
\psi(c_{(1)} \otimes \phi(c_{(2)})) &= (\sum_i a^i_\alpha \gamma({c_{(2)}}^\alpha \otimes c_i))_\beta \otimes {c_{(1)}}^\beta \\
&= \sum_i a^i_{\alpha\delta} \gamma({c_{(2)}}^\alpha \otimes c_i)_\beta \otimes {c_{(1)}}^{\delta\beta} && \text{(by (1))} \\
&= \sum_i a^i_\alpha \gamma({c^\alpha}_{(2)} \otimes c_i)_\beta \otimes {{c^\alpha}_{(1)}}^\beta && \text{(by (2))} \\
&= \sum_i a^i_\alpha \gamma(c^\alpha \otimes c_{i(1)}) \otimes c_{i(2)} && \text{(by (11))} \\
&= \sum_{i,j} (a^j a^i_\beta)_\alpha \gamma(c^\alpha \otimes c_j^\beta) \otimes c_i && \text{(by (15))} \\
&= \sum_{i,j} a^j_\alpha a^i_{\beta\delta} \gamma(c^{\alpha\delta} \otimes c_j^\beta) \otimes c_i && \text{(by (1))} \\
&= \sum_{i,j} a^j_\alpha \gamma(c^\alpha \otimes c_j) a^i \otimes c_i = \phi(c)\rho^A(1_A) && \text{(by (12))}
\end{aligned}$$

Using normalisation of γ as well as (15) one easily finds that $\sum_i a^i \phi(c_i) = 1_A$. Finally, take any $b \in B$, $c \in C$ and compute

$$\begin{aligned}
b_\alpha \phi(c^\alpha) &= \sum_i b_\alpha a^i_\beta \gamma(c^{\alpha\beta} \otimes c_i) = \sum_i (ba^i)_\alpha \gamma(c^\alpha \otimes c_i) && \text{(by (1))} \\
&= \sum_i (a^i b_\beta)_\alpha \gamma(c^\alpha \otimes c_i^\beta) = \sum_i a^i_\alpha b_{\beta\delta} \gamma(c^{\alpha\delta} \otimes c_i^\beta) && (b \in B,\ (1)) \\
&= \sum_i a^i_\alpha \gamma(c^\alpha \otimes c_i) b = \phi(c) b && \text{(by (12))}
\end{aligned}$$

Therefore ϕ satisfies all the conditions of Proposition 5.6 and, consequently, $B \hookrightarrow A$ is a split extension. □

Dually to Definition 5.1 we can consider

Definition 5.9. Let $(A,C)_\psi$ be an entwining structure. Any $\zeta \in \mathrm{Hom}(C, A\otimes A)$ such that the following diagrams

$$\begin{array}{ccccc} C\otimes A & \xrightarrow{\zeta\otimes A} & A\otimes A\otimes A & \xrightarrow{A\otimes\mu_A} & A\otimes A \\ \| & & & & \uparrow{\scriptstyle \mu_A\otimes A} \\ C\otimes A & \xrightarrow{\psi} & A\otimes C & \xrightarrow{A\otimes\zeta} & A\otimes A\otimes A \end{array} \tag{16}$$

$$\begin{array}{ccccc} C & \xrightarrow{\Delta_C} & C\otimes C & \xrightarrow{\zeta\otimes C} & A\otimes A\otimes C \\ \downarrow{\scriptstyle \Delta_C} & & & & \uparrow{\scriptstyle A\otimes\psi} \\ C\otimes C & \xrightarrow{C\otimes\zeta} & C\otimes A\otimes A & \xrightarrow{\psi\otimes A} & A\otimes C\otimes A \end{array} \tag{17}$$

commute is called a *cointegral map* in $(A,C)_\psi$. A cointegral map ζ is said to be *normalised*, if $\mu_A\circ\zeta = 1_A\circ\epsilon_C$.

Theorem 5.10. *The forgetful functor $\mathbf{M}_A^C(\psi)\to\mathbf{M}^C$ is separable if and only if there exists a normalised cointegral map in $(A,C)_\psi$.*

Proof. Consider an admissible morphism $(1_A, C) : (k,C)_\sigma \to (A,C)_\psi$ and apply Theorem 3.4(2). □

Example 5.11. Let $(A,C)_\psi$ be a canonical entwining structure associated to a copointed coalgebra-Galois extension $A(k)^B$ of k. Then the forgetful functor $\mathbf{M}_A^C(\psi)\to\mathbf{M}^C$ is separable.

In this case a normalised cointegral map is $\zeta = \mathrm{can}^{-1}\circ(1_A\otimes C)$.

6. Strongly separable coalgebra-Galois extensions

In this section we combine the results of previous two sections to determine when a coalgebra-Galois extension is a strongly separable extension. Such an extension was introduced in [14] in order to describe algebraic aspects of the Jones knot polynomial.

Definition 6.1. An extension of algebras $B\hookrightarrow A$ is called a *strongly separable extension* if it is a separable and split extension, and there exist a separation idempotent $u=\sum_i u_i\otimes u^i$, a conditional expectation $E : A\to B$ and a unit $\tau\in k$ such that for all $a\in A$,
(i) $\sum_i E(au_i)u^i = a\tau$
(ii) $\sum_i u_iE(u^ia) = a\tau$.

Proposition 6.2. *Let $A(B)^C$ be a coalgebra-Galois extension. If there exist a normalised integral $\mathfrak{z}=\sum_i a_i\otimes c_i$ and a normalised integral map $\gamma\in\mathrm{Hom}(C\otimes C, A)$ in the canonical entwining structure $(A,C)_\psi$, and a unit $\tau\in k$ such that*
(i) $\sum_i a_i 1_{A(0)}{}_\alpha\gamma(c_i^\alpha\otimes 1_{A(1)}) = \tau$,
(ii) $\sum_i a_{i(0)}\gamma(a_{i(1)}\otimes c_i) = \tau$,
then $B\hookrightarrow A$ is a strongly separable extension.

Proof. By Proposition 4.5, $B \hookrightarrow A$ is separable with $u = \sum_i u_i \otimes u^i = \mathrm{can}^{-1}(\mathfrak{z})$, while by Proposition 5.8, $B \hookrightarrow A$ is split with a conditional expectation $E : A \mapsto a_{(0)} 1_{A(0)_\alpha} \gamma(a_{(1)}{}^\alpha \otimes 1_{A(1)}) = (a1_{A(0)})_{(0)} \gamma((a1_{A(0)})_{(1)} \otimes 1_{A(1)})$. Take any $a \in A$ and compute:

$$\begin{aligned}
\sum_i E(au_i)u^i &= \sum_i E(u_i)u^i a && (u \text{ is an integral})\\
&= \sum_i u_{i(0)} 1_{A(0)_\alpha} \gamma(u_{i(1)}{}^\alpha \otimes 1_{A(1)}) u^i a\\
&= \sum_i u_{i(0)} 1_{A(0)_\alpha} u^i_{\beta\delta} \gamma(u_{i(1)}{}^{\alpha\delta} \otimes 1_{A(1)}{}^\beta) a && (\text{by } (12))\\
&= \sum_i u_{i(0)} (1_{A(0)} u^i_\beta)_\alpha \gamma(u_{i(1)}{}^\alpha \otimes 1_{A(1)}{}^\beta) a && (\text{by } (1))\\
&= \sum_i u_{i(0)} u^i{}_{(0)_\alpha} \gamma(u_{i(1)}{}^\alpha \otimes u^i{}_{(1)}) a && (A \in \mathbf{M}_A^C(\psi))\\
&= \sum_i (u_i u^i{}_{(0)})_{(0)} \gamma((u_i u^i{}_{(0)})_{(1)} \otimes u^i{}_{(1)}) a && (A \in \mathbf{M}_A^C(\psi))\\
&= \sum_i a_{i(0)} \gamma(a_{i(1)} \otimes c_i) a = \tau a && (\mathfrak{z} = \mathrm{can}(u))
\end{aligned}$$

Therefore the condition Definition 6.1(i) is satisfied. Furthermore

$$\begin{aligned}
\sum_i u_i E(u^i a) &= \sum_i a u_i E(u^i) && (u \text{ is an integral})\\
&= \sum_i a u_i u^i{}_{(0)} 1_{A(0)_\alpha} \gamma(u^i{}_{(1)}{}^\alpha \otimes 1_{A(1)})\\
&= \sum_i a a_i 1_{A(0)_\alpha} \gamma(c_i^\alpha \otimes 1_{A(1)}) = \tau a && (\mathfrak{z} = \mathrm{can}(u))
\end{aligned}$$

This proves Definition 6.1(ii) and thus completes the proof of the proposition. □

Proposition 6.3. *Let k be a field and let $A(B)^C$ be a coalgebra-Galois extension with both A and B finite dimensional. Suppose that A_B is free. Then $B \hookrightarrow A$ is a strongly separable extension if and only if there exists a normalised integral $\mathfrak{z} = \sum_i a_i \otimes c_i$ in the canonical entwining structure $(A, C)_\psi$, a map $\phi : C \to A$ satisfying conditions (i)–(iii) in Proposition 5.6, and a non-zero $\tau \in k$ such that*

$$\sum_i a_i \phi(c_i) = \tau. \tag{18}$$

Proof. By Proposition 4.5, $B \hookrightarrow A$ is separable with $u = \sum_i u_i \otimes u^i = \mathrm{can}^{-1}(\mathfrak{z})$, while by Proposition 5.6, $B \hookrightarrow A$ is split with a conditional expectation $E : a \mapsto a_{(0)}\phi(a_{(1)})$. By [11, Remark 1.4(d)], Definition 6.1(i) holds provided that condition Definition 6.1(ii) holds. Thus it suffices to prove that (18) is a sufficient and

necessary condition for Definition 6.1(ii). Take any $a \in A$ and compute:

$$\begin{aligned}\sum_i u_i E(u^i a) &= \sum_i a u_i E(u^i) \qquad\qquad (u \text{ is an integral})\\ &= \sum_i a u_i {u^i}_{(0)} \phi({u^i}_{(1)}) = \sum_i a a_i \phi(c_i).\end{aligned}$$

Therefore $\sum_i u_i E(u^i a) = \tau a$ if and only if (18) holds. □

References

[1] T. Brzeziński. Frobenius properties and Maschke-type theorems for entwined modules. *Preprint*, math/9806025, 1998. To appear in *Proc. Amer. Math. Soc.*

[2] T. Brzeziński. On modules associated to coalgebra-Galois extensions. *Preprint*, q-alg/9712023, 1997. To appear in *J. Algebra*.

[3] T. Brzeziński and P.M. Hajac. Coalgebra extensions and algebra coextensions of Galois type. *Preprint*, q-alg/9708010, 1997. To appear in *Commun. Algebra*.

[4] T. Brzeziński and S. Majid. Coalgebra bundles. *Commun. Math. Phys.*, 191:467–492, 1998.

[5] S. Caeneepel, B. Ion, G. Militaru and S. Zhu. Separable functors for the category of Doi-Hopf modules. Applications. *Adv. Math.* to appear.

[6] S. Caeneepel, G. Militaru and S. Zhu. Doi-Hopf modules, Yetter-Drinfel'd modules and Frobenius type properties. *Trans. Amer. Math. Soc.* , 349:4311–4342, 1997.

[7] S. Caeneepel and S. Raianu. Induction functors for the Doi-Koppinen unified Hopf modules. [in:] *Abelian Groups and Modules*, A. Facchini and C. Menini (eds.), Kluwer Academic Publishers, Dordrecht 1995, pp. 73–94.

[8] A. del Rio. Categorical methods in graded ring theory. Publ. Mat., 36:489–531, 1992.

[9] F. DeMeyer and E. Ingraham. *Separable Algebras Over Commutative Rings.* Lecture Notes in Mathematics 181, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

[10] Y. Doi. Unifying Hopf modules. *J. Algebra*, 153:373–385, 1992.

[11] D. Fischman, S. Montgomery and H.-J. Schneider Frobenius extensions of subalgebras of Hopf algebras. *Trans. Amer. Math. Soc.* 349:4857–4895, 1997.

[12] G. Hochschild. Relative homological algebra. *Trans. Amer. Math. Soc.* 82:246–269, 1956.

[13] K. Hirata and K. Sugano. On semisimple extensions and separable extensions over non commutative rings. *J. Math. Soc. Japan* 18:360–374, 1966.

[14] L. Kadison. The Jones polynomial and certain separable Frobenius extensions. *J. Algebra* 186:461–475, 1996.

[15] M. Koppinen. Variations on the smash product with applications to group-graded rings. *J. Pure Appl. Alg.* , 104:61–80, 1994.

[16] E.F. Müller and H.-J. Schneider. Quantum homogeneous spaces with faithfully flat module structure. *Preprint*, 1998.

[17] C. Năstăsescu, M. van den Bergh and F. van Oystaeyen. Separable functors applied to graded rings. *J. Algebra* 123:397–413, 1989.

[18] R. Pierce. *Associative algebras.* Springer-Verlag, New York/Berlin, 1982.

[19] P. Podleś. Quantum spheres. *Lett. Math. Phys.*, 14:193–202, 1987.

[20] M.D. Rafael. Separable functors revisited. *Commun. Algebra* 18:1445–1459, 1990.

[21] S.L. Woronowicz. Compact matrix pseudogroups. *Commun. Math. Phys.*, 111:613–665, 1987.

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